***Infinitely-long Hallway Problem**

***Rabbit Breeding Model**

***Chambered Nautilus**

***Pascal's Triangle**

***Tower of Hanoi**

***Handshake Exploration**

***Sum of 100 Numbers**

***Wage Choice Problem**

***Flying Birds Problem**

Infinite Hallway Puzzle

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Fibonacci Sequence

Starting with "0" and "1", add previous two numbers for next number in sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .

Pascal's Triangle

A triangle (pyramid) starting with "1" where each level below starts and ends with "1", and middle numbers are sums of the two numbers DIRECTLY above it.

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

Binomial Expansion using Pascal's Triangle

Expanding Binomials:

1) coefficients match Pascal's Triangle.
2) Power of 1st variable decreases.
3) Power of 2nd variable increases.
4) When subtracting, alternate positive and negative terms.

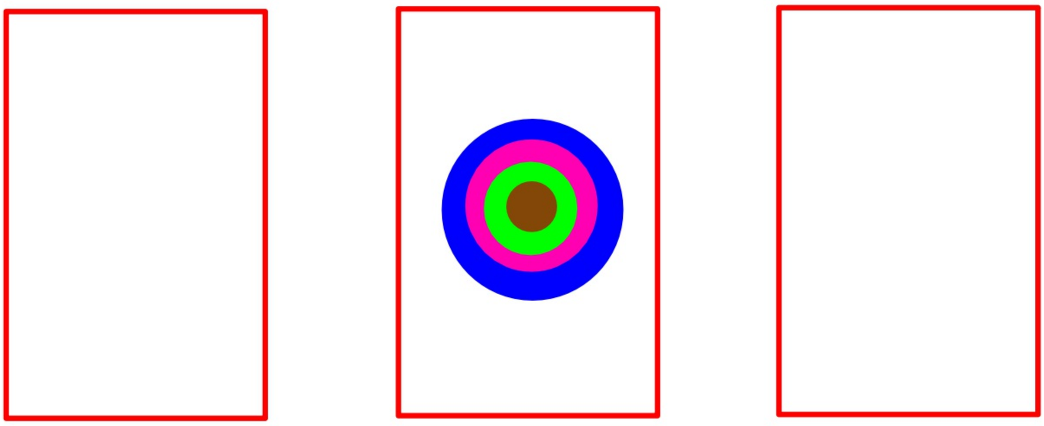
Row		Power
1	1	0
2	1 1	1
3	1 2 1	2
4	1 3 3 1	3
5	1 4 6 4 1	4
6	1 5 10 10 5 1	5
7	1 6 15 20 15 6 1	6

Examples:

$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

"Tower of Hanoi" Logic Puzzle



Goal: Move the entire tower to a different "mat" in the minimum number of moves.
Rule 1: Move ONLY the top piece from a mat.
Rule 2: Do not place any piece on top of a smaller piece.

Tower of Hanoi Solution Function

Levels	Moves
1	
2	
3	
4	
5	
6	
10	

Equation?

Sequence Formulas

Arithmetic Sequence:

$a_n = a_1 + d(n - 1)$

Geometric Sequence:

$a_n = a_1 r^{n-1}$

n = sequence number
a_n = nth number
a_1 = initial (first) number
d = common difference
r = common ratio

If you know 3, you can find the 4th.

Formulas for Sums of a Series:

Arithmetic Sum (finite):

$S_n = n(a_1 + a_n)/2$

Geometric Sum (finite):

$S_n = a_1(1 - r^n) / (1 - r)$ OR

$S_n = a_1(r^n - 1) / (r - 1)$

Geometric Sum (infinite); $|r| < 1$:

$S_\infty = a_1 / (1 - r)$

Sequence/Series Vocabulary:

Converge:
To come together

Diverge:
To go apart

Fixed point:
a point where that a sequence converges to

Closed Form:
The explicit form

Arithmetic Sequence Practice

Problem 1:
The first term of an arithmetic sequence is equal to 6 and the common difference is equal to 3. Find a formula for the n th term and the value of the 50 th term

Problem 2:
The first term of an arithmetic sequence is equal to 200 and the common difference is equal to -10. Find the value of the 20 th term

Problem 3:
An arithmetic sequence has a common difference equal to 10 and its 6 th term is equal to 52. Find its 15 th term.

Problem 4:
An arithmetic sequence has a its 5th term equal to 22 and its 15 th term equal to 62. Find its 100 th term

Problem 5:
Find the sum of all the integers from 1 to 1000.

Problem 6:
Find the sum of the first 50 even positive integers.

Arithmetic Sequence Practice

Find the next four terms in each arithmetic sequence.

1. $-1.1, 0.6, 2.3, \ldots$ **2.** $16, 13, 10, \ldots$ **3.** $p, p+2, p+4, \ldots$

For exercises 4–12, assume that each sequence or series is arithmetic.

4. Find the 24th term in the sequence for which $a_1 = -27$ and $d = 3$.

5. Find n for the sequence for which $a_n = 27$, $a_1 = -12$, and $d = 3$.

6. Find d for the sequence for which $a_1 = -12$ and $a_{23} = 32$.

7. What is the first term in the sequence for which $d = -3$ and $a_6 = 5$?

8. What is the first term in the sequence for which $d = -\frac{1}{3}$ and $a_7 = -3$?

9. Find the 6th term in the sequence $-3 + \sqrt{2}, 0, 3 - \sqrt{2}, \ldots$.

10. Find the 45th term in the sequence $-17, -11, -5, \ldots$.

11. Write a sequence that has three arithmetic means between 35 and 45.

12. Write a sequence that has two arithmetic means between -7 and 2.75.

13. Find the sum of the first 13 terms in the series $-5 + 1 + 7 + \cdots + 67$.

14. Find the sum of the first 62 terms in the series $-23 - 21.5 - 20 - \cdots$.

15. ***Auditorium Design*** Wakefield Auditorium has 26 rows, and the first row has 22 seats. The number of seats in each row increases by 4 as you move toward the back of the auditorium. What is the seating capacity of this auditorium?

Geometric Sequence Practice

Determine the common ratio and find the next three terms of each geometric sequence.

1. $-1, 2, -4, \ldots$

2. $-4, -3, -\frac{9}{4}, \ldots$

3. $12, -18, 27, \ldots$

For exercises 4–9, assume that each sequence or series is geometric.

4. Find the fifth term of the sequence $20, 0.2, 0.002, \ldots$.

5. Find the ninth term of the sequence $\sqrt{3}, -3, 3\sqrt{3}, \ldots$.

6. If $r = 2$ and $a_4 = 28$, find the first term of the sequence.

7. Find the first three terms of the sequence for which $a_4 = 8.4$ and $r = 4$.

8. Find the first three terms of the sequence for which $a_6 = \frac{1}{32}$ and $r = \frac{1}{2}$.

9. Write a sequence that has two geometric means between 2 and 0.25.

10. Write a sequence that has three geometric means between -32 and -2.

11. Find the sum of the first eight terms of the series $\frac{3}{4} + \frac{9}{20} + \frac{27}{100} + \cdots$.

12. Find the sum of the first 10 terms of the series $-3 + 12 - 48 + \cdots$.

13. ***Population Growth*** A city of 100,000 people is growing at a rate of 5.2% per year. Assuming this growth rate remains constant, estimate the population of the city 5 years from now.

Sigma Notation

$$\text{maximum value of } n \rightarrow \sum_{n=1}^{k} a_n \leftarrow \text{expression for general term}$$

starting value of n → (lower limit $n=1$)

↑ *index of summation*

Example 1 **Write each expression in expanded form and then find the sum.**

a. $\sum_{n=1}^{5} (n + 2)$

First, write the expression in expanded form.

$\sum_{n=1}^{5} (n + 2) = (1 + 2) + (2 + 2) + (3 + 2) + (4 + 2) + (5 + 2)$

Then, find the sum by simplifying the expanded form. $3 + 4 + 5 + 6 + 7 = 25$

b. $\sum_{m=1}^{\infty} 2\left(\frac{1}{4}\right)^m$

$$\sum_{m=1}^{\infty} 2\left(\frac{1}{4}\right)^m = 2\left(\frac{1}{4}\right)^1 + 2\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right)^3 + \cdots$$
$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$$

This is an infinite series. Use the formula $S = \frac{a_1}{1 - r}$.

$S = \frac{\frac{1}{2}}{1 - \frac{1}{4}}$ $\quad a_1 = \frac{1}{2}, r = \frac{1}{4}$

$S = \frac{2}{3}$

Example 2 **Express the series $26 + 37 + 50 + 65 + \cdots + 170$ using sigma notation.**

Notice that each term is one more than a perfect square. Thus, the nth term of the series is $n^2 + 1$. Since $5^2 + 1 = 26$ and $13^2 + 1 = 170$, the index of summation goes from $n = 5$ to $n = 13$.

Therefore, $26 + 37 + 50 + 65 + \cdots + 170 = \sum_{n=5}^{13} (n^2 + 1)$.

Sigma Notation Practice

Write each expression in expanded form and then find the sum.

1. $\sum_{n=3}^{5}(n^2 - 2^n)$

2. $\sum_{q=1}^{4}\frac{2}{q}$

3. $\sum_{t=1}^{5}t(t-1)$

4. $\sum_{t=0}^{3}(2t-3)$

5. $\sum_{c=2}^{5}(c-2)^2$

6. $\sum_{i=1}^{\infty}10\left(\frac{1}{2}\right)^i$

Express each series using sigma notation.

7. $3 + 6 + 9 + 12 + 15$

8. $6 + 24 + 120 + \cdots + 40{,}320$

9. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100}$

10. $24 + 19 + 14 + \cdots + (-1)$

Iteration and Iterates

Iteration: to compose a function with itself

x_0 = the initial (start) value; x_1 = the first iterate

Example 1:

$f(x) = 3x + 4$; $x_0 = 2$

$x_1 = 3(2) + 4 = 10$

$x_2 = 3(10) + 4 = 34$

$x_3 = 3(34) + 4 = 106$

Example 2:

$f(z) = 2z + 2i$; $z_0 = 1 - i$

$z_1 = 2(1 - i) + 2i = 2$

$z_2 = 2(2) + 2i = 4 + 2i$

$z_3 = 2(4 - i) + 2i = 8 + 6i$

Iteration Practice

Find the first four iterates of each function using the given initial value. If necessary, round your answers to the nearest hundredth.

1. $f(x) = x^2 + 4; x_0 = 1$

2. $f(x) = 3x + 5; x_0 = -1$

3. $f(x) = x^2 - 2; x_0 = -2$

4. $f(x) = x(2.5 - x); x_0 = 3$

Find the first three iterates of the function $f(z) = 2z - (3 + i)$ for each initial value.

5. $z_0 = i$

6. $z_0 = 3 - i$

7. $z_0 = 0.5 + i$

8. $z_0 = -2 - 5i$

Inductive Proof

A **proof** by **induction** is just like an ordinary **proof** in which every step must be justified. However it employs a neat trick which allows you to prove a statement about an arbitrary number n by first proving it is true when n is 1 and then assuming it is true for n=k and showing it is true for n=k+1.

Mathematical Induction

A method of proof called **mathematical induction** can be used to prove certain conjectures and formulas. The following example demonstrates the steps used in proving a summation formula by mathematical induction.

Prove that the sum of the first n positive even integers is $n(n + 1)$.

Here S_n is defined as $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

1. First, verify that S_n is valid for the first possible case, $n = 1$. Since the first positive even integer is 2 and $1(1 + 1) = 2$, the formula is valid for $n = 1$.

2. Then, assume that S_n is valid for $n = k$.

$S_k \Rightarrow 2 + 4 + 6 + \cdots + 2k = k(k + 1)$. *Replace n with k.*

Next, prove that S_n is also valid for $n = k + 1$.

$$S_{k+1} \Rightarrow 2 + 4 + 6 + \cdots + 2k + 2(k + 1)$$

$$= k(k + 1) + 2(k + 1) \quad \textit{Add 2(k + 1) to both sides.}$$

We can simplify the right side by adding $k(k + 1) + 2(k + 1)$.

$$S_{k+1} \Rightarrow 2 + 4 + 6 + \cdots + 2k + 2(k + 1)$$

$$= (k + 1)(k + 2) \quad \textit{(k + 1) is a common factor.}$$

If $k + 1$ is substituted into the original formula $(n(n + 1))$, the same result is obtained.

$(k + 1)[(k + 1) + 1]$ or $(k + 1)(k + 2)$

Thus, if the formula is valid for $n = k$, it is also valid for $n = k + 1$. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. That is, the formula for the sum of the first n positive even integers holds.

Jack 'n' Arnie

Arnold Palmer and Jack Nicklaus play golf together. After each hole, the loser owes the winner money according to the following rules:

1) One cent for first hole.
2) Amount doubles each hole.

Jack wins the first 17 holes;
Arnold wins 18th hole.

How much does each golfer win from the other?
Who owes who money and how much?

The Bouncing Basketball

A basketball drops from a height of 10 feet. Each time it hits the floor, it rebounds 2/3 of its previous height.

1) Write a sequence to show the height of the first 5 bounces.

2) What total vertical distance will the ball travel before it comes to rest?